Quest for Mathematics I (E2): Exercise sheet 2 solutions

1. (a)
$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{4n^2-1}=\frac{1}{2}\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)=\frac{1}{2}\lim_{N\to\infty}\left(1-\frac{1}{2N+1}\right)=\frac{1}{2}.$$
   (b)
$$\sum_{n=1}^{N}\frac{1}{n(n+2)}=\frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+2}\right)=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{N+1}-\frac{1}{N+2}\right)\to\frac{3}{4}.$$
   Hence
$$\sum_{n=1}^{\infty}\frac{1}{n(n+2)}=\frac{3}{4}.$$
   (c)
$$0.343434=0.34\sum_{n=0}^{\infty}(0.01)^n=\frac{0.34}{1-0.01}=\frac{34}{99}.$$

2. (a) We have that
$$\frac{a_{n+1}}{a_n}=\frac{x}{n+1}\le\frac{1}{2}$$
   for $n>2x-1$. Hence, the ratio test (comparison with a geometric series) tells us the series converges.

   (b) Similarly, we have that
$$\frac{a_{n+1}}{a_n}=2\left(\frac{n}{n+1}\right)^5\ge 2$$
   for all n, and so the ratio test tells us the series diverges.

   (c) Note that, for N even,
$$\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n}=\sum_{n=1}^{N/2}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)=\sum_{n=1}^{N/2}\frac{1}{2n(2n-1)}.$$
   Similarly, for N odd,
$$\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n}=\sum_{n=1}^{(N-1)/2}\frac{1}{2n(2n-1)}+\frac{1}{N}.$$
   In particular,
$$\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n}=\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n}=\sum_{n=1}^{\lfloor N/2\rfloor}\frac{1}{2n(2n-1)}+\frac{1}{2N}(1-(-1)^N).$$
   By the sandwich theorem,
$$\lim_{N\to\infty}\frac{1}{2N}(1-(-1)^N)=0.$$
   Moreover,
$$\lim_{N\to\infty}\sum_{n=1}^{\lfloor N/2\rfloor}\frac{1}{2n(2n-1)}\le\lim_{N\to\infty}\sum_{n=1}^{\lfloor N/2\rfloor}\frac{1}{(2n-1)^2}\le\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty,$$
   and so the series converges.

3. (a) This is a geometric series with initial value x and ratio $(1-x)^2$. Hence, if $x=0$, all the terms are zero, and so the sequence converges. Otherwise, the sequence converges if and only if $(1-x)^2 < 1$, i.e. $-1 < 1-x < 1$. Hence the interval I on which the series converges is given by $I = [0, 2)$.

   (b) We have that

$$f(x) = \begin{cases} 0 & \text{for } x = 0, \\ \frac{1}{2-x} & \text{for } x \in (0, 2). \end{cases}$$

   (c) The function f is continuous on $(0, 2)$, and has a right limit at $x = 0$ (but is not right continuous there).

4. The function is not defined where $x^3 - x = 0$, i.e. $x \in \{-1, 0, 1\}$. For x not in this set,

$$f(x) = \frac{x^3 - x^2}{x^3 - x} = \frac{x^2(x-1)}{x(x+1)(x-1)} = \frac{x}{x+1}.$$

   Thus, if we set $f(0) = 1$, $f(1) = \frac{1}{2}$, the function will be continuous at these points. For $x = -1$, f has an asymptote, and so no choice of $f(-1)$ will make the function continuous there.

5. The function is given by

$$f(x) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } 0 < |x| \le 1, \\ -1 & \text{for } |x| > 1. \end{cases}$$

   Points of discontinuity at $-1, 0, 1$. 0 is removable, ± 1 are jump discontinuities.